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Spherical Gravitational Collapse with Heat Flux and Cosmic Censorship

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Abstract

In this paper, we investigate the nature of the singularity in the spherically symmetrical, shear-free, gravitational collapse of a star with heat flux using a separable metric [1]. For any non-singular, regular, radial density profile for a star described by this metric, eq. (2.1), the singularity of the gravitational collapse is not naked *locally*. Our results here unequivocally support the Strong Cosmic Censorship Hypothesis.

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I. INTRODUCTION

The problem of whether the gravitational collapse leads, under general astrophysical circumstances, to a black hole or to a naked spacetime singularity - the Cosmic Censorship Hypothesis (CCH) - is one of the important problems of modern physics. However, this has proven to be a very difficult question to answer. It is therefore important for the proper mathematical formulation of this question that we study as many of the gravitational collapse situations as possible. It is expected that such studies would help sharpen the statement of the CCH.

In a recent work [1] we have studied spherically symmetric, separable metric spacetimes with energy/heat flux. The source matter in those spacetimes can satisfy **any** equation of state, in particular, it is allowed to be of the form $p = \alpha\rho$ where p and ρ are the pressure and the density of the matter and α is a constant. We consider here the formation of singularities in the gravitational collapse of a star described using those spacetimes.

The organization of this paper is as follows: In §II we outline the features of the metric of the shear-free, spherically symmetric, separable-metric spacetime with heat flux. This class of solutions is then analyzed in §III for the existence of the strong curvature spacetime singularity. In §IV we argue that the solutions can model a star with heat flow and an extended atmosphere. In §V we analyze the radial light cone equation for this class of metrics and show that a horizon always forms before the singularity. We show that for a regular, non-singular, density profile of a star, the singularity is not even naked locally. Finally, in §VI, we discuss the implications of our results for the Cosmic Censorship Hypothesis.

II. THE SPACETIME METRIC

The spacetime metric of *all* the spherically symmetric, shear-free, separable metric solutions with matter allowed to possess the equation of state $p = \alpha\rho$ is [1]

$$ds^2 = -\,y^2\,dt^2 \;+\; R^2\left[2\,(y')^2\,dr^2 \;+\; y^2\,(d\theta^2 + \sin^2(\theta)\,d\phi^2)\right] \tag{2.1}$$

where $y \equiv y(r)$, $R \equiv R(t)$ and an overhead prime has been used to denote the derivative with respect to the radial coordinate r. The coordinates (t, r, θ, ϕ) are comoving. Note that the spatial part of the metric (2.1) is, in general, non-flat.

The non-vanishing components of the Ricci tensor for the metric (2.1) are

$$R_{00} = -3\,\frac{\ddot{R}}{R} + \frac{1}{R^2} \tag{2.2}$$

$$R_{01} = 2\,\frac{\dot{R}}{R}\frac{y'}{y} \tag{2.3}$$

$$R_{11} = 2\,\frac{y'^2 R^2}{y^2}\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right) \tag{2.4}$$

$$R_{22} = R\dot{R}\left(\frac{\ddot{R}}{R} + 2\,\frac{\dot{R}}{R}\right) \tag{2.5}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.6}$$

The Ricci and the Kretschmann scalars of the metric (2.1) are

$$\mathcal{R} = \frac{6}{y^2}\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - \frac{1}{y^2 R^2} \tag{2.7}$$

$$\mathcal{K} \equiv R^{abcd}R_{abcd} = \frac{3 - 4\,\dot{R}^2 + 12\,\dot{R}^4}{y^4\,R^4} - \frac{8\,\ddot{R}}{y^4\,R^3} + \frac{12\,\ddot{R}^2}{y^4\,R^2} \tag{2.8}$$

Clearly, the metric and the spacetime are singular for $y = 0$ and/or $R = 0$.

The fluid four-velocity U^a and the fluid four-acceleration $\dot{U}_a = U_{a;b}U^b$ are given by

$$U^a = \frac{1}{y}\delta^a{}_0 \tag{2.9}$$

$$\dot{U}_a = (0, \frac{y'}{y}, 0, 0) \tag{2.10}$$

where a semicolon denotes a covariant derivative.

We observe that the R_{01} component of the Ricci tensor or, equivalently, the G_{01} component of the Einstein tensor, is non-vanishing. This implies the presence of non-vanishing energy-flux in the spacetime under consideration. The energy-flux can be physically interpreted in different possible ways. Firstly, there could be bulk motions of matter. Secondly, there could be "thermodynamical" heat flux arising from microphysical considerations of matter. For the purpose of a stellar application here, we shall consider the interpretation arising out of thermodynamical considerations of the stellar matter.

As a result, the energy-momentum tensor of the matter fluid is of the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} + q_a U_b + q_b U_a \tag{2.11}$$

where p is the pressure, ρ is the density of the fluid and $q^a = (0, q, 0, 0)$ is the radial heat flux four-vector. Note that both shear and rotation vanish for the metric (2.1).

The expansion, Θ, heat flux, q, density, ρ and pressure, p, are given by

$$\Theta = \, U^a{}_{;a} \, = \frac{3}{y}\frac{\dot{R}}{R} \tag{2.12}$$

$$q = \frac{-1}{y^2\, y'}\,\frac{\dot{R}}{R^3} \tag{2.13}$$

$$\rho = \frac{1}{y^2\, R^2}\left[\frac{1}{2} \,+\, 3\,\dot{R}^2\right] \tag{2.14}$$

$$p = \frac{1}{2y^2R^2} \;-\; \frac{\dot{R}^2}{y^2R^2} \;-\; \frac{2\ddot{R}}{y^2R} \tag{2.15}$$

Note that $\rho \propto 1/y^2$ and that the radial function is not fixed by the field equations. Hence, we can, for a non-singular density distribution, choose a nowhere vanishing radial function $y(r)$. Note, however, that for $y(r) = r$ or $y' = 1$, the initial density profile is singular at $r = 0$ [1].

From eq. (2.14) and (2.15), we obtain

$$\ddot{R} \;=\; \frac{y^2R}{6}\left[\frac{2}{y^2R^2} \;-\; (\rho + 3p)\right] \tag{2.16}$$

It is then seen from eq. (2.16) that the equation of state for the collapsing matter closes the system of ordinary differential equations obtainable from the field equations and leads to definite dynamics for the collapsing matter.

We note that, for the metric (2.1), the equation of state for the collapsing matter could change with the progress of the collapse. In other words, we emphasize that the metric (2.1), although initially obtained [1] for a specific equation of state of the form $p = \alpha\rho$ for α being a constant, admits *any* equation of state for the collapsing matter. In fact, the equation of state is an additional input of a physical nature that is to be supplied in order to determine the dynamics of collapsing matter.

Further, the change in the equation of state of the stellar matter could "halt" the collapse and may result in a semi-stable or stable matter configuration. However, since our interest here is in the end-result of the unstoppable gravitational collapse, we, in what follows, assume that the gravitational collapse is unstoppable. Hence, our further results below apply to "unstoppable" gravitational collapse.

We also emphasize that in the present description we could begin with an initial stellar density configuration that is non-singular for all r. Of course, the singularity will develop as a result of the time evolution when the physical radius of the collapsing stellar shell vanishes at some moment of the time, *ie.* when $R = 0$ at some $t = t_o$, say. Further, from symmetry considerations, such a singularity will, of course, be located at $r = 0$.

III. THE SPACETIME SINGULARITY

From the physical point of view, a singularity could be termed *strong* if a test body is crushed to zero volume as it approaches the singularity. Hence a sufficient condition for the singularity to be a strong curvature type as defined by Tipler *et al* [3] is:

> For at least one null geodesic with an affine parameter s, with $s = 0$ at the singularity, the following holds

$$\lim_{s\to 0} s^2 R_{ab} K^a K^b > 0 \tag{3.1}$$

where K^a is the tangent vector to the geodesic in question and R_{ab} is the Ricci tensor.

If this condition is satisfied then the spacetime is not extendible [4]. Such spacetimes with strong curvature singularities are to be considered in view of the CCH.

The geodesics for the metric (2.1) are easily obtainable [5] by considering the metric as a Lagrangian:

$$2\mathcal{L} = -y^2\tilde{t}^2 + R^2\left[2y'^2\tilde{r}^2 + y^2\tilde{\theta}^2 + y^2\sin^2(\theta)\,\tilde{\phi}^2\right] \tag{3.2}$$

where an overhead tilde denotes a derivative with respect to the affine parameter s along the geodesic and $2\mathcal{L}$ has values $+1, -1$ and 0 for the spacelike, timelike and null geodesics, respectively.

The geodesic equations are then obtainable as

$$\frac{d}{ds}\left(Ry^2\tilde{t}\right) = -\,2\mathcal{L}\,\frac{dR}{dt} \tag{3.3}$$

$$\frac{d}{ds}\left(R^2yy'\tilde{r}\right) = \mathcal{L} \tag{3.4}$$

$$\frac{d}{ds}\left[\left(R^2y^2\tilde{\theta}\right)^2 + k^2\cot^2\theta\right] = 0 \tag{3.5}$$

$$\frac{d}{ds}\left(R^2y^2\sin^2\theta\tilde{\phi}\right) = 0 \tag{3.6}$$

where k is a constant of integration from the ϕ-equation.

Let us then consider the radial null geodesics of the metric (2.1) for the purpose of the above condition since it requires the existence of at least one null geodesic for its purpose. Now, the future-directed tangent to the *radial null* geodesic has the following components

$$K^t \equiv \frac{dt}{ds} = \frac{K_1}{y^2R} \tag{3.7}$$

$$K^r \equiv \frac{dr}{ds} = \frac{K_2}{2yy'R^2} \tag{3.8}$$

where K_1, K_2 are constants. Since K^a is null, we have ${K_2}^2 \,=\, 2\,{K_1}^2$.

We now have

$$R_{ab}K^aK^b \;=\; \left(-3\frac{\ddot{R}}{R} + \frac{1}{R^2}\right)\frac{K_1^2}{y^4R^2} \;+\; \left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right)\frac{K_2^2}{2y^4R^2} \;+\; \frac{2\dot{R}K_1K_2}{y^4R^4} \tag{3.9}$$

$$= \frac{K_1^2}{y^2R^2}\,(\rho\,+\,p) \;-\; \frac{2K_1K_2}{y^2}\,y'\,R\,q \tag{3.10}$$

We can now, after some manipulations based on the energy conditions, in particular, $(p+\rho)^2 > 4\,q^2$ (see Appendix), show that

$$\lim_{s\to 0}\; s^2R_{ab}K^aK^b \;\longrightarrow\; \infty \tag{3.11}$$

Note that we have $s=0$ at the singularity $R=0$.

Hence, the singularity of the metric (2.1) is a strong curvature singularity in the sense of Tipler [3] *et al.* The singularity in a spherical gravitational collapse with heat flow considered here is then a strong curvature singularity always.

This completes our description of the shear-free, spherically symmetric, separable metric solutions of the Einstein field equations that represent an inhomogeneous distribution of imperfect matter with heat flux. Different energy conditions can also be satisfied for the metric (2.1). (See Appendix A. This spacetime is also of interest to cosmology [2].)

IV. STELLAR DENSITY PROFILE

Any initial density configuration of the star must be non-singular for all r. For example, by noting that $\rho \propto 1/y^2$, the radial function [1] could be chosen as

$$y^2 = 1 + \exp\left(\frac{r - r_o}{L}\right) \tag{4.1}$$

where r_o is the boundary radius and L is the thickness of the boundary layer. The corresponding star has density that decreases with r and at the boundary radius, r_o, there is a sharp fall in density within the layer L. Note that the density does not vanish outside the star but can be very low as compared to that inside the star. In the case of our sample

profile, eq. (4.1), the star has an atmosphere of exponentially decreasing density. Essentially, the star is embedded in a cosmological spacetime of some sort and the spacetime is not necessarily asymptotically flat. The presence of an extended atmosphere enveloping the star has been observed with real stars, for example, with the Sun. *We emphasize that we do not assume the profile (4.1) to obtain the results presented here.*

Also, the "exterior" of the star as considered here needs to be specified. If we consider the exterior of the star as being described by the Vaidya spacetime then we obtain the matching condition [6] as:

$$p_\Sigma = \sqrt{2}\, y\, y'|_\Sigma \tag{4.2}$$

where Σ is the timelike hyper-surface across which the interior and the exterior spacetimes are matched. This condition restricts the allowed forms of the temporal function $R(t)$ at the stellar boundary. However, for our purposes, it is not necessary to consider these restrictions.

Moreover, note that $q \propto -\dot{R}/y'$. During the collapse, we must have $\dot{R} < 0$ (from (2.12)). Hence, with our assumption that $y' > 0$, the heat flux is positive, *ie.* heat flows from high temperature regions to low temperature regions. Our model of a star with heat flow therefore has a higher temperature at the center. The temperature decreases as we move away from the stellar center. This is what we expect in any realistic model of a star. In our model, the density will be a decreasing function of the radial coordinate r when $y' > 0$ and this is what we shall assume.

Of course, $y(r)$ can be constrained by other considerations. For example, the thermal stability of the stellar structure, heat transport equation etc. will, in a manner similar to that of the Newtonian theory, yield some physical constraints on the radial distribution of density and, hence, on $y(r)$. In fact, it is these considerations that must be used to obtain the detailed stellar model in a manner similar to that of the Newtonian theory. (However, we wish to point out here that these are not relevant to the subject of the present paper mainly because the present considerations apply to all the forms of the function $y(r)$.)

We should also keep in mind that we require the spacetime manifold to be locally flat

in the neighborhood of the origin, $r = 0$, of the coordinate system, as it should be for any point of the spacetime manifold. In our case, a small circle of coordinate radius ϵ with center at the origin has circumference of $2\pi\epsilon$. On the other hand, the circle has the proper radius $\sqrt{2}\, y'\, \epsilon$. Then, requiring that the ratio of the circumference to the proper radius of the circle to be 2π in the neighborhood of the origin, we obtain the condition $y'|_{r=0} \approx 1/\sqrt{2}$. This condition must be imposed on $y(r)$ to obtain a realistic density profile of a star.

V. OUTGOING RADIAL NULL GEODESICS

The radial null cone equation is

$$\frac{dt}{dr} = \pm \sqrt{2}\, \frac{y'}{y}\, R \tag{5.1}$$

where we have to choose the positive sign for the radially outgoing null geodesics.

The physical radius of the collapsing shell of matter is $R_{Ph} = yR$. The collapsing, spherically symmetric shell of a star must have some physical radius R_{Ph} at $t = t_o$. The physical radius becomes zero at $t = 0$ signifying that the shell in question has collapsed to a singularity. The singularity will, of course, be at the radial coordinate $r = 0$ due to symmetry considerations. The singularity of the metric is then at $t = 0$ and $r = 0$.

Further, we could choose $y(r) = r$, that is, $y' = 1$. However, as remarked earlier, this choice corresponds to a singular initial density profile and, hence, is unacceptable to us since a normal star does not show such a density profile. Therefore, we use the non-singular density profile henceforth.

To determine whether the singularity is globally visible or not, we need to look at the convergence of the principal null congruence and determine the condition for the formation of the outermost trapped surface. Now, the outgoing principal null vector for the metric (2.1) is

$$\ell^a \;=\; \frac{1}{\sqrt{2}} \left[\frac{1}{y}\, \delta^a{}_t \;+\; \frac{1}{\sqrt{2}\, y'\, R}\, \delta^a{}_r \right] \tag{5.2}$$

To test for the existence of a trapped surface, we set the expansion rate for ℓ^a to zero and look for a real solution that is positive, R_{ah}. This solution corresponds to the apparent horizon - the time history of the marginally trapped surface of the spacetime geometry of the metric (2.1).

Moreover, since we are dealing with separable metric functions, we expect to obtain a condition only on the time-dependent part of the metric, that is, $R(t)$. The condition is

$$\dot{R} \;=\; -\frac{1}{\sqrt{2}} \tag{5.3}$$

The trapped surface then always covers the singularity since $\dot{R}$ must attain the above value as it becomes (negatively) unbounded with the progress of the collapse. Hence, the spacetime singularity is *not* globally visible in all of the cases being considered below.

When we consider a non-singular density profile, we shall have

$$\lim_{r\to 0} \frac{y'}{y} = \ell_o$$

say, where ℓ_o is the appropriate positive and finite limiting value. Now, for the geodesic tangent to be uniquely defined and to exist at the singular point, $t = 0$ and $r = 0$, the following must hold so that an outgoing, future-directed photon trajectory exists at the singularity:

$$\lim_{\substack{r\to 0\\ t\to 0}} \frac{t}{r} = \lim_{\substack{r\to 0\\ t\to 0}} \frac{dt}{dr} = X_o \tag{5.4}$$

where X_o is required to be *real* and *positive.* As we approach the singular point, we have,

$$X_o = \sqrt{2} \lim_{\substack{r\to 0\\ t\to 0}} \frac{y'}{y} R(t) \;=\; \sqrt{2} \lim_{t\to 0} \ell_o R(t) \tag{5.5}$$

Clearly, the limit in question implies that $X_o = 0$. Hence, there does not exist a real and positive tangent to the null geodesic at the singularity if the initial density profile corresponds to a density distribution that is non-singular.

Therefore, no null geodesics emerge from the singularity and communicate to an external observer. Then, for any non-singular density profile of a star with heat flux the singularity of collapse is not visible to any observer.

VI. DISCUSSION

A genuine spacetime singularity is a singular point of the very structure of the spacetime. The usual laws of classical physics break down at such a point of the spacetime manifold. It is therefore impossible to account for the effects of the singularity if it were to causally affect its exterior. Further, the usual laws of quantum physics, relativistic or non-relativistic, also fail to hold at such a spacetime location. This is mainly because such laws have been obtained using the background spacetime arena or the notions of space and time of the background metric. Such notions are unavailable at the singular point of the spacetime manifold. Unless the quantum theory of geometry or gravitation, whatever that means, is invoked the problem of accounting for the usual physical phenomena remains if any genuine spacetime singularity were to causally affect its exterior. In other words, the *visible* or *naked* spacetime singularity becomes a genuine problem for the classical theories of physics.

It is well-known [7] that solutions with *visible* or *naked* spacetime singularities can be obtained in the General Theory of Relativity as a field theory of gravitation. Constructing such a spacetime geometry and equating the corresponding Einstein tensor with the energy-momentum tensor always produces a solution of the field equations of general relativity. However, it remains to show why such solutions cannot be considered *physical* in some definite sense of the term, *ie.* we require some principle on the basis of which visible spacetime singularities can be considered to be unphysical. This is the problem of the Cosmic Censorship Hypothesis (CCH). Moreover, since the classical theories can be considered to be complete theories, in some appropriate sense, such a principle may be expected to be classical in nature or content.

Thus one of the most important but as-yet unsolved problems of the classical theory of general relativity is undoubtedly that of the CCH. In fact, this problem is also important for the fundamental theories of physics since it directly concerns the nature of the gravitational field in the strong field limit. It is the same hypothesis that is at the helm of the very existence of Black Holes and hence Black Hole Physics, Astrophysics and Thermodynamics

as well [7]. As a result it is certainly important to either prove or disprove the expectation that this principle is of a classical nature.

Unfortunately, it has been extremely difficult to arrive at the precise and generally agreeable formulation of the CCH let alone prove it. This is primarily because of the conceptual nature of the problem itself. The statement of the CCH is that *the spacetime singularity be not visible to any legitimate physical observer in a generic, realistic, physical situation such as gravitational collapse* [8]. We emphasize that this is simply an expectation based on our inability to treat a genuine spacetime singularity in a classical theory. It necessarily follows that if any genuine spacetime singularity were to causally affect its exterior then the external observer would be unable to account for the physical phenomena in its vicinity. The classical predictability would then be in jeopardy if the singularity were *visible* to the external observer.

As regards the statement of the hypothesis, the real issue then is about making the terms such as *generic*, *physical* or *realistic* precise so as to make some mathematically provable or disprovable statement for this principle of Cosmic Censorship. For example, we could demand that the collapsing matter satisfy some energy conditions/equation of state, that the spacetime not possess any special symmetry, that it should be stable to perturbations and so on. However, it has proven to be rather difficult to translate such demands into any generally agreeable mathematically precise statements.

In the absence of any precise formulation of the CCH and hence its proof, we can, at best, look for examples of naked or visible singularities. From such examples, we can hope to sharpen the meaning of various terms in the statement of the CCH.

In the recent past, many workers [9] have followed this approach. However, we note that none of these examples can be considered to be a genuine counter-example to the CCH since all can be considered to be *special* in some or the other sense. Nonetheless, it is true that such examples can be used to provide some definite meaning to the statement of the censorship hypothesis.

To illustrate the above point, we note that the locally naked singularity of the Vaidya-de

Sitter metric can be shown [10] to be the same as that of the Vaidya metric. Then, we can argue that the asymptotic flatness of the solution does not manifest, in some sense, in the occurrence or not of the naked spacetime singularity, *ie.* the asymptotic observer has no role to play in the formulation of the censorship hypothesis. Therefore, this example unequivocally supports the formulation of the strong CCH due to Penrose [7]. Essentially, the Strong CCH demands that the space-time singularity be not visible to any observer.

In this paper, we considered the shear-free collapse of a spherical star with heat flux using the space-time of the metric (2.1). Moreover, our model, based on the metric (2.1), can accommodate the realistic feature that the star can possess an extended atmosphere similar to what is observed with the real stars. Our results show that the singularity resulting in the stellar collapse modelled using the metric (2.1) is not visible to *any observer* if the initial density profile of the star is non-singular similar to what is observed with real stars. The strong cosmic censorship hypothesis states that the singularity be not visible to any observer. Hence, our results are in complete accord with the Strong CCH for the non-singular density profile of the stellar model considered here.

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APPENDIX A:

In this appendix, we discuss the energy conditions as obtainable for the considered spacetime. (For an elegant analysis and further details of energy conditions applicable to imperfect fluids, see [11].)

We recall the definitions of the energy conditions for the sake of completeness:

- *Weak energy condition*: $T_{ab}W^aW^b \geq 0$ where W^a is a timelike, future-directed 4-vector. It implies that the energy density as measured by any observer is *positive*.

- *Dominant energy condition*: $F_a = T_{ab}W^b$ must be future-directed, timelike or null for any timelike, future-directed W^a. It implies that the speed of energy flow of matter must be less than the speed of light for every observer.

- *Strong energy condition*: $2\,T_{ab}W^aW^b + T \geq 0$ for any timelike, future-directed, unit 4-vector W^a where T is the trace of T_{ab}.

We may note here that the Dominant energy condition implies the Weak energy condition, that the Strong energy condition can be violated in the presence of negative pressure, that is, when $T_{ab}W^aW^b$ is negative and that the investigation of energy conditions is an algebraic problem that leads to a search of the roots of a polynomial of degree 4 in a four dimensional spacetime.

From [11], we also recall here the corollary for the shear-free or dynamic-viscosity-free, spherically symmetric spacetimes. We note that the metric (2.1) describes a shear-free spacetime.

Fluid that undergoes shear-free motion or has vanishing coefficient of dynamic viscosity satisfies

- the Weak energy condition iff

$$\rho + p \geq 2q \tag{A1}$$

$$\rho - p + \triangle > 0 \tag{A2}$$

- the Dominant energy condition iff, in addition to (A1), we have

$$\rho - p \geq 0 \qquad\qquad \rho - 3p + \triangle \geq 0 \tag{A3}$$

- the Strong energy condition iff, in addition to (A1), we have

$$2p + \triangle \geq 0 \tag{A4}$$

where $\triangle = \sqrt{(p+\rho)^2 - 4q^2}$.

In general, these conditions are to be imposed on the spacetime under consideration. However, it is easy to verify that these conditions are satisfied for the spacetime of the metric (2.1) when $0 \leq \alpha \leq 1$ for the simple equation of state $p = \alpha\rho$.

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